19/08/26

Finally back to work! :)

More Gauges:

- Uniform density gauge

  - Freedom in time slicing allows us to set $\delta\rho = 0$
  - $\delta g_{ij} = a^2(1-2\zeta)\delta_{ij}$ is the main scalar perturbation

- Comoving Gauge

  - We enforce $v + B = 0$
  - $\delta g_{ij} = a^2(1-2\mathcal{R})\delta_{ij}$
  - a convenient gauge for the computation of the inflationary quantum fluctuations

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6.1.3 - Conservation Eq^{n}s.

We now want to derive the evolution of these perturbations.

The evolution of $\left\{ \begin{array}{l} \text{the metric} \\ \text{matter perturbations} \end{array} \right.$ is governed by / follows from $\left\{ \begin{array}{r} G_{\mu\nu} = 8\pi G T_{\mu\nu} \\ \nabla^\mu T_{\mu\nu} = 0 \end{array} \right.$

Expanding these equations to linear order in the pert. will give us linearized equations for both 'elements' of interest.

In the following, we fix the gauge to be the Newtonian gauge.
Recall in this gauge

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\delta_{ij} \end{pmatrix}$$

Let us first work to linearize $\nabla^\mu T_{\mu\nu} = 0$.
Sps. that there is no transfer of energy or momentum between each component $T^{(a)}_{\mu\nu}$.
$\Rightarrow$ also $\nabla^\mu T^{(a)}_{\mu\nu} = 0$ and thus any result applies to each species separately.

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More explicitly,

$$\nabla_\mu T^\mu{}_\nu = 0$$
$$= \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha$$

We present the example of computing Γ^0_{00} as indicative of the process...

Recall $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} (\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\beta\rho} - \partial_\rho g_{\beta\gamma})$

Inverting the metric:

$$g^{\mu\nu} = a^{-2} \begin{pmatrix} -(1-2\Psi) & 0 \\ 0 & (1+2\Phi)\delta^{ij} \end{pmatrix}$$

using the fact that $(1+2\Psi)^{-1} \approx (1-2\Psi)$

Then using $\Gamma^0_{00} = \frac{1}{2} g^{0\rho} (\partial_0 g_{0\rho} + \partial_0 g_{0\rho} - \partial_\rho g_{00})$ and that $g^{0i} = g^{i0} = 0$

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2} g^{00} (\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) \\
&= \frac{1}{2} g^{00} (\partial_0 g_{00}) \\
&= \frac{1}{2} a^{-2} (1-2\Psi)\, \partial_0 [a^2 (1+2\Psi)] \\
&= \frac{1}{2} a^{-2} (1-2\Psi) \left(a'a(1+2\Psi) + a'a(1+2\Psi) + a^2(2\Psi') \right) \\
&= a^{-2}(1-2\Psi)\left(a'a(1+2\Psi) + a^2\Psi'\right)
\end{aligned}$$

Working to linear order,

$$\begin{aligned}
&= \frac{a'}{a} + \Psi' \\
&= \mathcal{H} + \Psi'
\end{aligned}$$

Omitting the derivations (for now) for the other symbols, we summarise the results:

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + \Psi' \\
\Gamma^0_{i0} &= \partial_i \Psi \\
\Gamma^i_{00} &= \partial^i \Psi \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} - [\Phi' + 2\mathcal{H}(\Phi + \Psi)]\delta_{ij} \\
\Gamma^i_{j0} &= (\mathcal{H} - \Phi')\delta^i_j \\
\Gamma^i_{jk} &= -2\delta^i{}_{(j}\partial_{k)}\Phi + \delta_{jk}\partial^i\Phi
\end{aligned}$$

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Recall from yesterday.

$$\nabla_\mu T^\mu{}_\nu = 0$$
$$= \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha \qquad (*)$$

Let us consider the $\nu = 0$ component of $(*)$,

$$\partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 + \underbrace{\Gamma^\mu_{\mu i} T^i{}_0}_{(1)} - \Gamma^0_{00} T^0{}_0 \underbrace{- \Gamma^0_{i0} T^i{}_0}_{(2)} \underbrace{- \Gamma^i_{00} T^0{}_i}_{(3)} - \Gamma^i_{j0} T^j{}_i = 0$$

(1): $\Gamma^\mu_{\mu i} T^i{}_0 = \Gamma^0_{0i} T^i{}_0 + \Gamma^j_{ji} T^i{}_0$
$$= (\partial_i \Psi)(-[\bar\rho + \bar P]v^i) + (-2\delta^j{}_{(j}\partial_{k)}\Phi + \delta_{ji}\partial^j\Phi)(-[\bar\rho + \bar P]v^i)$$

(2): $-\Gamma^0_{i0} T^i{}_0 = -(\partial_i \Psi)(-[\bar\rho + \bar P]v^i)$

(3): $-\Gamma^i_{00} T^0{}_i = -(\partial^i \Psi)(T^0{}_i)$

Clearly (1), (2), (3) are $O(2)$ terms and thus discarded.

$$\Rightarrow \partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^0_{00} T^0{}_0 - \Gamma^i_{j0} T^j{}_i = 0$$

From previous results,

$$-(\bar\rho + \delta\rho)' - \partial_i q^i - (\mathcal{H} + \Psi' + 3\mathcal{H} - 3\Phi')(\bar\rho + \delta\rho) + (\mathcal{H} + \Psi')(\bar\rho + \delta\rho) - (\mathcal{H} - \Phi')\delta^i_j(\bar P + \delta P)\delta^j_i = 0$$

$$\Rightarrow \bar\rho' + \delta\rho' + \partial_i q^i + 3\mathcal{H}(\bar\rho + \delta\rho) - 3\bar\rho\Phi' + 3\mathcal{H}(\bar P + \delta P) - 3\bar P\Phi' = 0$$

Zeroth - order : $\boxed{\bar\rho' = -3\mathcal{H}(\bar\rho + \bar P)}$ ~ the conservation of energy in the homogeneous background

First - order : $\boxed{\delta\rho' = -3\mathcal{H}(\delta\rho + \delta P) - \partial_i q^i + 3\Phi'(\bar\rho + \bar P)}$ ~ the continuity eq^n of $\delta\rho$

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Let us now consider the $\nu = i$ component of $(*)$,

$$\partial_0 T^0{}_i + \partial_j T^j{}_i + \Gamma^\mu_{\mu 0} T^0{}_i + \Gamma^\mu_{\mu j} T^j{}_i - \Gamma^0_{0i} T^0{}_0 - \Gamma^0_{ji} T^j{}_0 - \Gamma^j_{0i} T^0{}_j - \Gamma^j_{ki} T^k{}_j = 0$$

Recall that we can write $q^i = (\bar{\rho} + \bar{P}) v^i$. Thus $T^0{}_i = q_i$ and $T^i{}_0 = -q^i$ so

$$q_i' + \partial_j \left[(\bar{P} + \delta P)\delta^j_i + \Pi^j{}_i \right] + 4\mathcal{H} q_i + (\partial_j \Psi - 3\partial_j \Phi)\bar{P}\delta^j_i + \partial_i \Psi \bar{\rho} + \mathcal{H}\delta_{ji} q^j - \mathcal{H}\delta^j_i q_j - \left(-2\delta^j{}_{(k}\partial_{i)}\Phi + \delta_{ki}\partial^j \Phi\right)\bar{P}\delta^k_j = 0$$

$$\Rightarrow \boxed{q_i' = -4\mathcal{H} q_i - (\bar{\rho} + \bar{P})\partial_i \Psi - \partial_i \delta P - \partial^j \Pi_{ij}} \quad \sim \text{ Euler eq}^{\underline{n}}$$

Let us now observe these equations somewhat closer.

Continuity Eq$^{\underline{n}}$: $\delta\rho' = \underbrace{-3\mathcal{H}(\delta\rho + \delta P)}_{(1)} \underbrace{- \partial_i q^i}_{(2)} + \underbrace{3\Phi'(\bar{\rho} + \bar{P})}_{(3)}$

" continuity eq$^{\underline{n}}$ describing the evolution of the density perturbations "

(1) - dilution due to background expansion

(2) - accounts for local fluid flow

(3) - purely relativistic, corresponds to the density changes caused by perturbations to the local expansion rate

$\rightsquigarrow$ $(1-\Phi)a$ can be thought of as the " local scale factor " in the spatial part of the metric in the Newtonian gauge.

Let us substitute $\delta\rho = \bar{\rho}\delta$ and $q^i = (\bar{\rho} + \bar{P})v^i$,

$$\Rightarrow \boxed{\delta' = -\left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(\nabla \cdot \vec{v} - 3\Phi') - 3\mathcal{H}\left(\frac{\delta P}{\delta \rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta} \quad (\dagger)$$

Which gives us an evolution eq$^{\underline{n}}$ for the density contrast.

Euler Eq$^{\underline{n}}$: $q_i' = \underbrace{-4\mathcal{H}q_i}_{(1)} \underbrace{- (\bar{\rho} + \bar{P})\partial_i \Psi}_{(2)} \underbrace{- \partial_i \delta P}_{(3)} \underbrace{- \partial^j \Pi_{ij}}_{(4)}$

(1) - enforces the expected scaling of q $(q \propto a^{-4})$

(2) - force due to gravity

(3) - force due to pressure

(4) - force due to anisotropic stress

Substituting $q_i = (\bar{\rho} + \bar{P})v_i$,

$$\Rightarrow \boxed{v_i' = -\left(\mathcal{H} + \frac{\bar{P}'}{\bar{\rho} + \bar{P}}\right)v_i - \frac{1}{\bar{\rho} + \bar{P}}(\partial_i \delta P + \partial^j \Pi_{ij}) - \partial_i \Psi} \quad (\S)$$

Which gives us an evolution equation for the bulk velocit

Let us now evaluate (†) and (‡) for some specific, special cases:

Matter: We consider a non-relativistic fluid with $P_m = 0$ and $\Pi^{ij}_m = 0$

$$\Rightarrow \quad \delta'_m = -\nabla\cdot\vec{v}_m + 3\Phi'$$
$$\vec{v}'_m = -\mathcal{H}\vec{v}_m - \nabla\Psi$$

friction

$$\Rightarrow \quad \delta''_m + \mathcal{H}\delta'_m = \nabla^2\Psi + 3(\Phi'' + \mathcal{H}\Phi')$$

gravity

Radiation: Consider a relativistic perfect fluid (i.e. radiation with no viscosity) with $P_r = \frac{1}{3}\rho_r$ and $\Pi^{ij}_r = 0$

$$\Rightarrow \quad \delta'_r = -\frac{4}{3}\nabla\cdot\vec{v}_r + 4\Phi'$$
$$\vec{v}_r = -\frac{1}{4}\nabla\delta_r - \nabla\Psi$$

$$\Rightarrow \quad \delta''_r - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Psi + 4\Phi''$$

If one wanted to work with (†) and (‡) in Fourier space we make the following replacements:

$$\partial_i \to ik_i \;, \quad \underbrace{v_i \to i\hat{k}_i v \;, \quad \Pi_{ij} \to -(\bar{\rho}+\bar{P})\hat{k}_{\langle i}\hat{k}_{j\rangle}\Pi}_{} \qquad \left(\hat{k}_{\langle i}\hat{k}_{j\rangle} \equiv \hat{k}_i\hat{k}_j - \tfrac{1}{3}\delta_{ij}\right)$$

Note: absorption of $|k|$ and rescaling means these are not quite the F.T of $v(\eta,\vec{x})$ and $\Pi(\eta,\vec{x})$

$$\Rightarrow \quad \delta' = \left(1 + \frac{\bar{P}}{\bar{\rho}}\right)(kv + 3\Phi') - 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta$$

$$v' = -\left(\mathcal{H} + \frac{\bar{P}}{\bar{\rho}+\bar{P}}\right)v - \frac{1}{\bar{\rho}+\bar{P}}k\delta P + \frac{2}{3}k\Pi - k\Psi$$

For now I will only include additional details on the use of perfect and barotropic fluids

- Perfect fluids are characterised by strong interactions which keep the pressure isotropic $\Pi = 0$
- For a barotropic fluid, with $P = P(\rho)$, we can write $\delta P = c_s^2\delta\rho$
- If $\delta P = c_s^2\delta\rho = \frac{\bar{P}'}{\bar{\rho}'}\delta\rho$ the perturbations are called adiabatic, which I may include more details later.

for barotropic

$$\left(c_s^2 = \frac{dP}{d\rho} = \frac{\bar{P}'}{\bar{\rho}'}\right)$$